TRINITY COLLEGE

# Year 12 Methods Units 3,4
# Test 4 2020

Section 1 Calculator Free

**Logarithms, Log Calculus & Continuous Random Variables**

**STUDENT'S NAME** MARKING KEY [KRISZYK]

**DATE:** Monday 3rd August **TIME:** 25 minutes **MARKS:** 26

**INSTRUCTIONS:**

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

1. (5 marks)

For each of the following state whether it is the probability density function of a continuous random variable. Justify your answer.

(a) $f(x) = \begin{cases} 0.2x + 0.1\,, & 1 \le x \le 3 \\ 0\,, & elsewhere \end{cases}$ [3]

$$\int_1^3 0.2x + 0.1\,dx = \left[\frac{0.2x^2}{2} + 0.1x\right]_1^3 = \left[0.1\left(x^2+x\right)\right]_1^3$$

$$= \left[0.1(12)\right] - \left[0.1(2)\right]$$

$$= 1 \quad \therefore \text{CRV.}$$

(b) [2]

| $x$ | $-1$ | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X = $x$) | 0.2 | 0.1 | 0.4 | 0.3 |

No, the data is discrete.

2. (8 marks)

Differentiate the following. Do not simplify.

(a) $\overset{u}{x^2} \overset{v}{\ln(4x^2)}$ [2]

$$\frac{d}{dx}\left(x^2 \ln(4x^2)\right)$$

$$= 2x\left[\ln(4x^2)\right] + \frac{2}{x}(x^2)$$

$u' = 2x$

$v' = \frac{8x}{4x^2} = \frac{2}{x}$

(b) $\ln\left(\frac{e^{3x}}{\cos(4x+1)}\right)$ [3]

$$= \ln\left[e^{3x}\right] - \ln\left[\cos(4x+1)\right]$$

$$= \frac{3e^{3x}}{e^{3x}} - \frac{-4\sin(4x+1)}{\cos(4x+1)}$$

(c) $\log_2 x^e$ [3]

$$= \frac{\ln x^e}{\ln 2}$$

$$= \frac{d}{dx}\left(\frac{\ln x^e}{\ln 2}\right)$$

$$= \frac{e}{\ln(2)x}$$

3. (3 marks)

Solve $\ln(x-3)-\ln 2=3$, giving your answer as an exact value.

$$\ln(x-3) - \ln 2 = 3$$

$$\ln\left[\frac{(x-3)}{2}\right] = 3$$

$$e^3 = \frac{x-3}{2}$$

$$2e^3 = x-3$$

$$x = 2e^3+3$$

4. (7 marks)

(a) Determine the simplified exact value of k, if X is a continuous random variable with a probability density function given by: [4]

$$f(x) = \begin{cases} \frac{2}{x-2} & \text{for } 4 \le x \le k \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_4^k \frac{2}{x-2}\,dx = 1$$

$$\left[ 2\ln(x-2) \right]_4^k = 1$$

$$2\ln(k-2) - 2\ln(4-2) = 1$$

$$2\ln(k-2) - 2\ln(2) = 1$$

$$\ln\left[\frac{k-2}{2}\right] = \frac{1}{2}$$

$$\frac{k-2}{2} = e^{\frac{1}{2}}$$

$$k = 2e^{1/2} + 2$$

(b) Determine the cumulative distribution function for $f(x)$ and use it to calculate the exact value for $P(X < 4.5)$ . [3]

$$P(X \le x) = \begin{cases} 0 \text{ for } x < 4 \\ 2\ln\left(\frac{x-2}{2}\right) \text{ for } 4 \le x \le 2e^{1/2}+2 \\ 1 \text{ for } x > 2e^{1/2}+2 \end{cases}$$

$$P(X < 4.5) = 2\ln\left(\frac{4.5-2}{2}\right)$$

$$= 2\ln\left(\frac{5}{4}\right)$$

5. (3 marks)

Determine the anti-derivative of $\dfrac{\cos(4x)}{7+\sin(4x)}$.

$$f'(x) = \frac{\cos(4x)}{7+\sin(4x)}$$

$$f(x) = \frac{1}{4}\int \frac{4\cos(4x)}{7+\sin(4x)}\,dx$$

$$= \frac{1}{4}\ln|7+\sin(4x)| + c$$

TRINITY COLLEGE

# Year 12 Methods Units 3,4
# Test 4 2020

Section 2 Calculator Assumed

**Logarithms, Log Calculus & Continuous Random Variables**

**STUDENT'S NAME** MARKING KEY [KRISZYK]

**DATE:** Monday 3rd August **TIME:** 25 minutes **MARKS:** 33

**INSTRUCTIONS:**

Standard Items: Pens, pencils, drawing templates, eraser

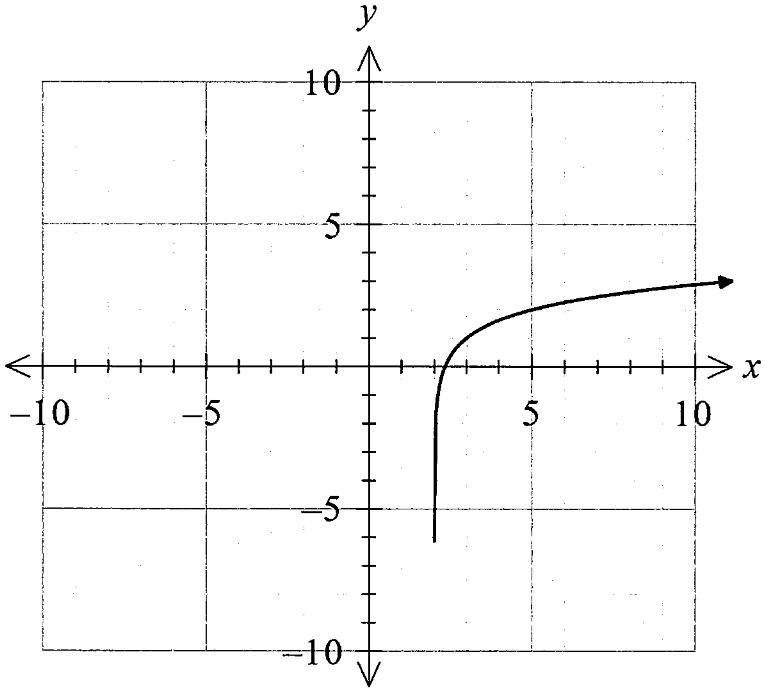
Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

6. (3 marks)

Determine the equation of the function shown below.




$y = \log_3(x-2) + 1$

7. (9 marks)

The time, t, in minutes, taken to shave using a hand razor, was found to satisfy a probability density function, defined by

$$f(t) = \begin{cases} kt\,e^{-\frac{t^2}{8}} & \text{for } t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Show that k has a unique value of 0.25 [3]

$$\int_0^\infty kte^{\frac{-t^2}{8}}\,dt = 1 \quad ✓$$

$$k\left[-4e^{-t^2/8}\right]_0^\infty = 1$$

$$k\left[[-4e^{\infty}] - [-4e^{0}]\right] = 1$$

$$k[0+4] = 1$$

$$k = \frac{1}{4}$$

must show these lines for marks 2 and 3 ✓✓

(b) Determine the probability that using a hand razor to shave will take more than three minutes. Give your answer correct to four decimal places. [2]

$$P(T>3) = \int_3^\infty 0.25t\,e^{-t^2/8}\,dt$$

$$= 0.3247$$

(c) Determine the mean and standard deviation of the hand razor shaving times, in minutes, correct to two decimal places. [2]

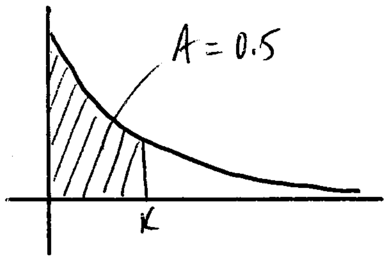
$$E(T) = 2.51$$

$$Var(T) = 1.7168$$

$$\therefore \quad SD(T) = \sqrt{1.7168}$$

$$= 1.31$$

(d) Determine the median time, in minutes, for hand razor shaving correct to three decimal places. [2]



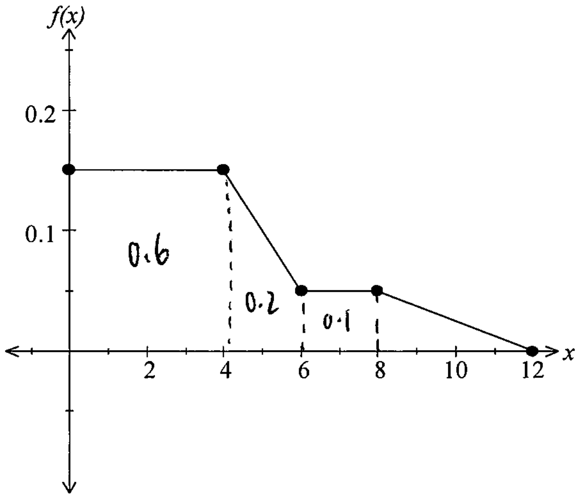

$$P(T<k) = 0.5$$

$$\int_0^k 0.25te^{-t^2/8}\,dt = 0.5$$

$k = 2.355$ minutes.

8. (5 marks)

The continuous random variable X has the probability density function shown below.




Calculate:

(a) $P(X < 0.6)$ $0.15 \times 0.6 = 0.09$ [1]

(b) $P(X \geq 7)$ [1]

$1 - \left[0.6 + \left(0.1 + \frac{1}{2}(2) \times 0.1\right) + 0.05\right]$

$= 0.15$

(c) $P(X \geq 7 \mid X > 2)$ [2]

$\frac{0.15}{1-(2 \times 0.15)} = \frac{0.15}{0.7} = \frac{3}{14}$

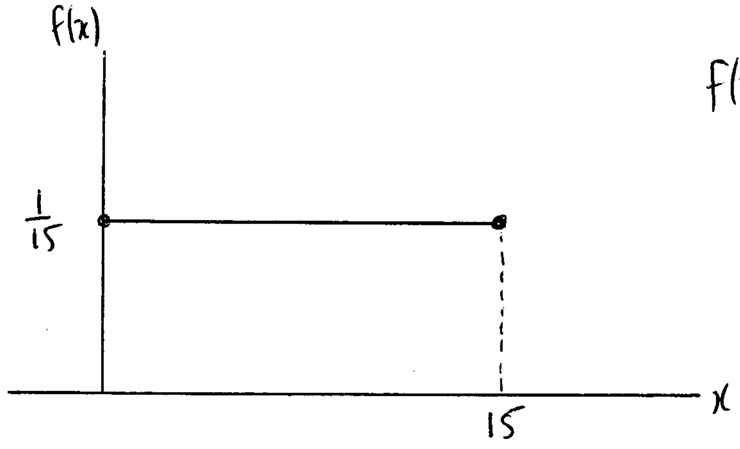
(d) $P(X = 5 \mid X > 2)$ [1]

$= 0$

9. (8 marks)

Alex catches the train every day (Monday to Friday) to school. His arrival time at the train station is uniformly distributed between 7.45 am and 8.00 am. Let the random variable X be the number of minutes Alex arrives at the train station after 7.45 am.

(a) Write down the probability density function for X and sketch its graph. [2]




$$f(x) = \begin{cases} \frac{1}{15} & \text{for } 0 \leq x \leq 15 \\ 0 & \text{for all other } x \end{cases}$$

(b) Determine the probability that Alex arrives at the train station before 7.55 am. [1]

$$P(X < 11) = \frac{10}{15}$$

(c) Determine the probability that Alex arrives at the train station before 7.55 am if he arrives later than 7.50 am. [2]

$$P(x < 10 \mid x > 5) = \frac{5}{10}$$
$$= \frac{1}{2}$$

(d) If Alex arrives at the train station after 7.55 am he would miss the 7.55 am train and need to wait for the 8.05 am train. In doing so he would be late for school. Assuming Alex's arrival time at the train station each day is independent, determine the probability that Alex is late for school at least once each week. [3]

Let $Y$ = days Alex is late

$\therefore Y \sim B(5, \frac{1}{3})$

$P(\text{late}) = \frac{5}{15} = \frac{1}{3}$

$P(Y \geq 1) = 0.8683$

10. (4 marks)

The decibel scale for sound, measured in decibels (dB), is defined as: $D = 20\log_{10}\left(\frac{P}{P_{ref}}\right)$,

where $P$ is the pressure of the sound being measured and $P_{ref}$ is a fixed reference pressure.

(a) What is the decibel measure for a sound with a pressure of $2P_{ref}$ ? [1]

$$D = 20 \log (2)$$
$$= 6.02$$

(b) The sound produced by a symphony orchestra measures 120 dB, while that of a rock concert measures 150 dB. How many times greater is the sound pressure of the rock concert than that of the orchestra? [3]

$$120 = 20 \log \left(\frac{P_o}{P_{ref}}\right) \qquad 150 = 20 \log \left(\frac{P_R}{P_{ref}}\right)$$

$$6 = \log \left(\frac{P_o}{P_{ref}}\right) \qquad 7.5 = \log \left(\frac{P_R}{P_{ref}}\right)$$

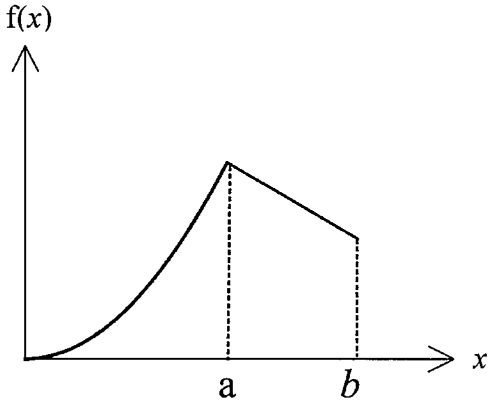
$$\therefore \quad P_o = 10^6 \qquad P_R = 10^{7.5}$$

$\therefore$ Rock concert has $\frac{10^{7.5}}{10^6} = 10^{1.5}$ times greater sound pressure.

11. (4 marks)

The probability density function for the continuous random variable $X$ is defined and graphed below.

$$f(x) = \begin{cases} \dfrac{x^2}{4} & 0 \le x \le a \\ \dfrac{4-x}{2} & a \le x \le b \\ 0 & elsewhere \end{cases}$$




Determine the exact values of $a$ and $b$.

$$\frac{x^2}{4} = \frac{4-x}{2}$$

$$x = -4 \text{ or } 2$$

$$\therefore \underline{\underline{a = 2}}$$

$$\int_0^2 \frac{x^2}{4}\,dx + \int_2^b \frac{4-x}{2}\,dx = 1$$

$$b = \frac{-2\sqrt{6}}{3} + 4 \quad \text{or} \quad \cancel{\frac{2\sqrt{6}}{4} - 4}$$

Reject as $f(x) > 0$.